ON ABRUPT ONSET OF STEADY FLOWS IN HYDRODYNAMICS

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This is a study of the transition from laminar steady motion to turbulent, in systems described by equations of the hydrodynamic type. We describe the method of finding periodic steady motions of small amplitude. The latter is expressed in explicit form in terms of the parameters of the system near the borderline separating the regions of smooth and abrupt transition.

1. It is well known that in the transition from the laminar state to the turbulent in a number of systems, a motion is set up with a definite frequency and wave vector when slight supercriticalness takes place (as an example we may take convection between parallel plates [1], the flow of a liquid between rotating cylinders [2 to 4], the strata in a gaseous discharge [5 and 6]. helical instability in a gaseous discharge and semi-conductors [7 to 10]; a motion, periodic with respect to time, arises also in the flow past rigid bodies [11]). The frequency and the wave number, and also the form of the oscillation, can be approximately determined from linear theory; to find the amplitude Q, however, it is necessary to take into account nonlinear effects.

It is shown below that the equation for the square of the modulus of the amplitude $q = QQ^*$ of a steady periodic motion has the form (for small q)

$$\frac{dq}{dt} = 2q(\gamma_0 + aq + bq^2 + \ldots) \equiv 2q\gamma \qquad (1.1)$$

The phase of the amplitude Q (and consequently also the phase of the steady solution) is arbitrary. The coefficients γ_0, a, b, ... are functions of the parameters of the system λ (the temperature, the geometric dimensions, the external magnetic field and so on); γ_0 is the growth rate given by linear theory, whilst the second and third terms in γ are related to the inclusion of nonlinear effects.

The critical parameters λ_* are defined by Equation $\gamma_0(\lambda_*) = 0$; the equilibrium state $q = 0$ is unstable when $\gamma_0(\lambda) > 0$.

Let the system be such that $a(\lambda_*) \neq 0$ for any values of λ_*. If $a(\lambda_*) < 0$, then with an increase of supercriticalness $\Lambda = \lambda - \lambda_*$ the amplitude of the steady motion continuously increases from zero (the "smooth" behavior); in this case from Equation $\gamma = 0$ we obtain $q \approx -(a^{-1} d\gamma_0 / d\lambda)_* \Lambda$. If $a(\lambda_*) > 0$, then as λ passes through the critical value λ_* the amplitude changes from zero up to a certain finite value by a jump [4, 6, 9 and 10] (the "abrupt" behavior); in this case the amplitude, in general, is not small for small supercriticalness, whilst the motion itself can have the irregular character of developed turbulent motion.

2. Let us consider systems for which, for certain values of the parameters λ_* , Equations $\gamma_0 = a = 0$ are satisfied. In such systems [2 to 10], depending on the values of λ_* , both smooth [3, 6 and 8] and abrupt transitions [4, 6, 9 and 10] of steady motion are possible.

Suppose that when $\lambda = \lambda_*$ the relations $\gamma_0 = a = 0$ and $b \neq 0$ are satisfied. Then for sufficiently small Λ the quantities γ_0 and a are small, whilst $b \neq 0$. The ratio of the quantities γ_0 and a is arbitrary, in so far as it depends upon the direction of the vector Λ . Accordingly, in finding solutions q of Equation $\gamma = 0$ we must regard the quantities γ_0 and a as independent small parameters. It is easy to find the solution q in the particular cases $a = 0$ or $\gamma_0 = 0$. In the first case q has the form of a series in powers of $\sqrt{\gamma_0}$, and in the second case in powers of a (moreover, there is the solution $q = 0$). In the general case the solution is sought in the form of a series $q = q_1 + q_2 + \ldots$, in which $q_n / q_{n-1} \to 0$ as $\Lambda \to 0$. In view of the nonsinglevaluedness of the choice of $q_n = q_n(\gamma_0, a)$ it is essential to require that the quantities $q_n(\gamma_0, 0)$ and $q_n(0, a)$ coincide with the nth terms in the expansions for q in powers of $\sqrt{\gamma_0}$ and a , respectively. In what follows we shall consider only the quantity

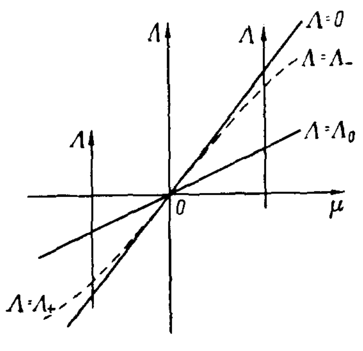

Fig. 1

$$q \approx q_1 = \frac{-a \pm \sqrt{a^2 - 4\gamma_0 b}}{2b} \qquad (2.1)$$

Let us select any two parameters λ and μ ; and let us fix the remaining parameters so that the curves $a(\lambda, \mu) = 0$ and $\gamma_0(\lambda, \mu) = 0$ intersect (Fig.1). We shall reckon λ and μ from the point of intersection and for definiteness we shall take the region $\gamma_0 > 0$ as located above the curve $\gamma_0 = 0$, and the region $a > 0$ above the curve $a = 0$. As the parameter λ varies, oscillations arise which are smooth if $\mu < 0$ and abrupt if $\mu > 0$

For small values of $|\mu|$ and $|\Lambda|$ (here $\Lambda = \lambda - \lambda_*$ is a scalar) we can assume that $\gamma = \gamma' \Lambda, \quad a = a'(\Lambda - \Lambda_0), \quad \Lambda_0 = c\mu,$ where c

and b and the derivatives γ', a' are taken at $\lambda = \mu = 0$. In the case shown in Fig.1 we have $\gamma' > 0$, $a' > 0$, $c < 0$. Expression (2.1) takes the form

$$q = A\,(\Lambda - \Lambda_0) \pm (A^2\,(\Lambda - \Lambda_0)^2 + B\Lambda)^{1/2} \qquad \left(A = \frac{-a'}{2b},\ B = \frac{-\gamma'}{b}\right) \qquad (2.2)$$

3. Suppose that $b < 0$ in Equation (2.2); then A and B are positive (this case is considered in [13]). If Λ varies along the straight line $\mu = \text{const} > 0$ (with $\Lambda_0 = c\mu < 0$), then when $\Lambda = +0$ the amplitude changes by a jump from zero to the value $q_0 = -2A\Lambda_0 \sim \mu$. If now Λ decreases, then for a certain value $\Lambda = \Lambda_-$, determined by Equation

$$A^2\,(\Lambda_- - \Lambda_0)^2 + B\Lambda_- = 0,$$

there occurs a drop in amplitude from the value $q_- = A\,(\Lambda_- - \Lambda_0)$ to zero (Fig.2). Since $|\Lambda_0| \sim \mu$ is a small quantity, then

$$\Lambda_- \approx -(A\Lambda_0)^2 / B = -(Ac\mu)^2 / B \sim \mu^2;$$

Since $|\Lambda_-| \ll |\Lambda_0|$ for small μ, then when $|\Lambda|$ is not large, Equation (2.2) can be put in the form

$$q = q_-\,(1 \pm \sqrt{1 - \Lambda / \Lambda_-}), \qquad q_- \approx -A\Lambda_0 \sim \mu,\ \Lambda_- \sim \mu^2 \qquad (3.1)$$
$$(|\Lambda| \leqslant |\Lambda_-|,\quad q_- > 0,\quad \Lambda_- < 0)$$

In the region $\Lambda_- < \Lambda < 0$ there exist two steady solutions (3.1) and $q = 0$. By means of Equation (1.1) we can see that the motion corresponding to solution (3.1) is stable (as observed experimentally), when the root is taken with the positive sign.

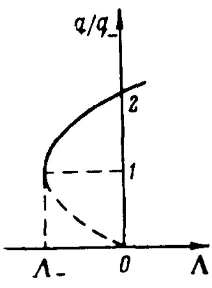

Fig. 2

In periodic motion of a medium any quantity χ (fluid velocity, temperature, charge density, etc.) varies periodically. Moreover, as is shown below, the harmonic X_ν of the periodic quantity χ obeys the relation $|X_\nu| \sim |Q|^{|\nu|}$. Hence it follows that in steady motion with small amplitude Q the form of the oscillation is close to sinusoidal [3,6,15 to 17). Accordingly, in the case of smooth transition and in the case considered above of abrupt transition ($a = \gamma_0 = 0$, $b < 0$ when $\lambda = \lambda_*$) the quantities' χ vary according to a sinusoidal law for small supercriticalness.

4. Let us consider the case $b > 0$; then A and $B < 0$. If Λ varies along the straight line $\mu = \text{const} > 0$, then when $\Lambda = +0$ the amplitude changes by a jump from zero to a certain large quantity. Moreover, the motion can at once acquire the irregular character of developed turbulent motion. If, however, a periodic motion is set up, then the oscillations have the form of relaxational oscillations, which are similar to discontinuous, and not to sinusoidal ones. The case $\mu > 0$ has not been successfully treated quantitatively.

Suppose that Λ varies along the straight line $\mu < 0$ (with $\Lambda_0 = c\mu > 0$, see Fig.1). Then as Λ varies from 0 to $\Lambda_+ \sim \mu^2$ the amplitude q varies from 0 to $q_+ \approx -A\Lambda_0 \sim \mu$ (Fig.3). On transition through the value Λ_+ the amplitude changes by a jump from the value q_+ to a certain large value, whilst the steady motion, close to sinusoidal when $q = q_+$, can acquire the irregular character of turbulent motion. If now Λ decreases, then for a certain $\Lambda = \Lambda_-$ there occurs a drop in the amplitude from a certain (in general large) value q_- to zero. A possible form of the dependence $q = q(\Lambda)$ is portrayed in Fig.3 (for the case when the motion with large amplitude remains periodic: this occurs, for example, in the case of strata [6] . In contrast to the case $b < 0$, when $\mu \to 0$ the quantities Λ_- and q_- do not vanish.

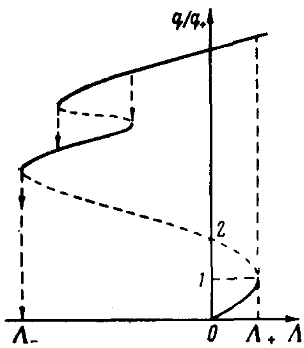

Fig. 3

When Λ and μ are not large, Expression (2.2) can be represented in the form

$$q = q_+ (1 \pm \sqrt{1 - \Lambda / \Lambda_+}), \qquad q_+ \sim \mu, \qquad \Lambda_+ \sim \mu^2 \tag{4.1}$$
$$(|\Lambda| \leqslant |\Lambda_+|, \quad q_+ > 0, \quad \Lambda_+ > 0)$$

The solution (4.1), in which the root is taken with the plus sign, is unstable.

We note that with increase of supercriticalness the amplitude of the stable solutions (continuous curves in Figs. 2 and 3) increases, whilst the amplitude of the unstable solutions (broken curves in Figs. 2 and 3) decreases.

5. Let us denote by q_* the unstable solutions (3.1) and (4.1). Suppose that for a certain value Λ the system was in a steady state $q < q_*$. If we impose on the system an external perturbation (variable e.m.f. in the external electric field, an impulse in a magnetic field, etc.) then for an amplitude of perturbation χ' exceeding the value

$$X_*' \sim \sqrt{q_*} \tag{5.1}$$

the system passes into the steady state $q > q_*$ and remains in it after removal of the external perturbation (this effect has been studied qualitatively in experiments [4 and 6]). If, however, $\chi' < \chi_*'$, then after removal of the perturbation the system again passes to the steady state $q < q_*$. The relation (5.1), in which q_* is taken from (3.1), passes for small Λ into the relationship

$$X_*' \sim \sqrt{-\Lambda}, \qquad \Lambda \to -0$$

which holds good also in the general case of abrupt transition, when $a(\lambda_*)$ is positive and not small [15].

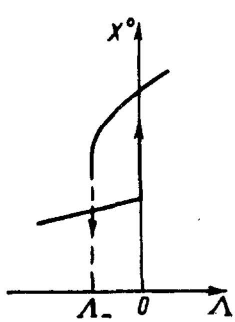

Fig. 4

As the amplitude of the steady motion varies, changes occur in the frequency ω and the mean value χ^0 (zeroth harmonic) of any observed quantity (mean temperature, magnetic induction, direct components of the current and so on); the corresponding dependences, as shown below, have the form

$$X^\circ = \chi + X_1 q + \ldots$$
$$\omega = \Omega_0 + \Omega_1 q + \ldots \tag{5.2}$$

if the steady motion is periodic in space, then the analogous relation $k = k_0 + k_1 q + \ldots$ holds for the wave number. The coefficients of the powers of q in (5.2) are analytic functions of Λ; the quantity χ corresponds to the equilibrium state $q = 0$. The values of Ω_0 and k_0 are determined from linear theory.

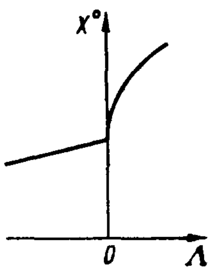

Fig. 5

From (5.2) it follows that corresponding with the corners (for smooth transitions) and jumps (for abrupt transitions) in the quantity $q(\Lambda)$, there are corners and jumps in the quantities X°, ω, k (such corners [8] and jumps [6, 9 and 10] are found experimentally). The form of the dependence $X^\circ(\Lambda)$ when $X_1(\lambda_*) > 0$ is shown in Fig.4 ($b < 0$, $\mu > 0$), Fig.5 ($b < 0$, $\mu = 0$), and Fig.6 ($b > 0$, $\mu < 0$). If in the case corresponding to Fig.4 we denote by $X_\sim$ the variable part of any quantity $X_\sim$ and by ΔX° the difference between the value of X° in the presence of the disturbances and in their absence for a fixed value of Λ, then it is not difficult to obtain [13] (5.2) from (3.1), and for sufficiently small μ it becomes

$$(X_\sim^2)_0 \,/\, (X_\sim^2)_- = (\Delta X^\circ)_0 \,/\, (\Delta X^\circ)_- = q_0 \,/\, q_- = 2 \tag{5.3}$$

The results presented above relate to steady motions of small amplitude, periodic with respect to time and (or) space. Such motions arise as a result of the development of growing perturbations, periodic with respect to time and (or) space. If, however, the perturbations growing in a slight supercriticalness are not periodic in time, nor in space, then these properties still pertain to the steady motion of small amplitude (such a situation is possible, for example, in the case of flow in a bounded space, caused by the motion of a boundary [12]). Such a motion is defined completely, in contrast to periodic steady flows, which are determined only to within an arbitrary phase. The expressions for the amplitude of such a motion are obtained if we replace in the nonlinear increment γ and in Equations (3.1), (4.1), (5.1), (5.3), the quantity $q = QQ^*$ by the positive amplitude Q. There is interest in finding experimentally in such systems [2 to 10] the points dividing the regions of smooth and abrupt transition, and verifying near such points the relations (3.1) and (5.1) to (5.3) in the case $b < 0$ and the relations (4.1) and (5.1) in the case $b > 0$.

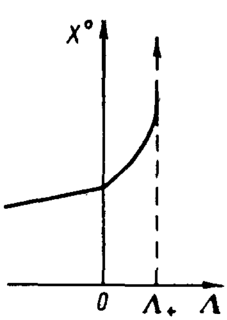

Fig. 6

6. It is shown below how to obtain Expressions (1.1) and (5.2) for γ, ω, k and X°. The equations of hydrodynamics have the form

$$F^i(X^j, \partial/\partial t, \nabla, r, \lambda) = 0 \qquad (i, j = 1, \ldots, N) \tag{6.1}$$

Here X are unknown quantities, λ are parameters of the system, r are spatial coordinates, ∇ are spatial differential operators; time does not appear in Equations (6.1) explicitly. The functions F are single-valued and analytic with respect to their arguments; wath respect to the differential operators they are polynomials.

Besides Equations (6.1), the quantities X have to satisfy, in general, inhomogeneous boundary conditions

$$U_j^i X^j = A^i \tag{6.2}$$

if the vector r belongs to the surface $S(r) = 0$ (here and in what follows where there are two identical indices, one of which is a subscript and the other a superscript, summation from 1 to N is to be understood). The quantities A depend upon r and λ, whilst the quantities U depend on the same arguments as the functions F. It can be assumed, however, that the quantities U do not depend upon X, i.e. that the conditions (6.2) are linear with respect to X; if this is not the case, then it is necessary to denote all the terms in (6.2) which are nonlinear with respect to X by X^j $(j > N)$, and to regard them as supplementary unknowns. Moreover, it can be taken that U does not depend upon $\partial/\partial t$; if this is not so, then it is necessary to denote all the derivatives with respect to t by $X^j (j > N)$ and regard these as supplementary unknowns.

In what follows the indices i and j of the quantities X, U and the others will be dropped; then X can be regarded as a vector, whilst U is a matrix operator, acting on X.

The equilibrium solution $X = \chi$ does not depend upon time and satisfies Equation

$$F_0 = 0, \qquad U\chi = A \tag{6.3}$$

Here F_0 is obtained from F by setting $\partial/\partial t = 0$.

When considering systems which are unbounded in space it is necessary to distinguish two cases. In the case of systems of the first type the equilibrium solution depends upon all the Cartesian coordinates (x, y, z) (flow past a body). Disturbances to equilibrium X_1 have the form

$$X_1 = QX_{11}e^{i\theta}, \qquad \theta = \omega t \tag{6.4}$$

Here Q is a constant of proportionality, whilst the functions $X_{11}(r)$ vanish [11] when $r \to \infty$. The frequencies $\omega = \Omega - i\gamma$ form a discrete spectrum; for slight supercriticalness only one characteristic perturbation grows, whilst for greater supercriticalness other perturbations can grow also. For slight supercriticalness the steady motion of small amplitude is always periodic with respect to time [11].

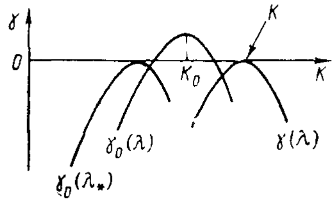

Fig. 7

In the case of systems of the second type the equilibrium solution does not depend upon one [2 to 10] or several [1] of the Cartesian coordinates. In this case the functions X_{11} in (6.4) depend upon those same coordinates as the equilibrium solution χ. The dependence of the perturbations upon the remaining coordinates is included in exponential factors (for example, in the case of unbounded systems [2 to 10] with cylindrical geometry $\chi = \chi(r)$ and $\theta = \omega t - m\varphi - kz$, where m is an integer and r, φ, z are cylindrical coordinates).

In this case even for slight supercriticalness there exists an infinite set of increasing perturbations with different wave numbers (Fig.7). It is not obvious that the interaction of these perturbations will always lead to the establishment of periodic motion, as occurs in the systems [1 to 10]. Apparently cases are possible where the steady motion of small amplitude, passing into equilibrium as $\Lambda \to +0$, consists of a continuous spectrum of waves, i.e. it is turbulent.

First of all let us consider systems of the second type; for definiteness we shall have in mind the systems [2 to 10] with cylindrical geometry. In this case the steady periodic solution of the problem (6.1), (6.2) has the form

$$X = \sum_{\nu=-\infty}^{\infty} X_\nu e^{i\nu\theta}, \quad \theta = \omega t - m\varphi - kz; \quad X_\nu = X_\nu(r), \quad X_{-\nu} = X_\nu^* \quad (6.5)$$

Here r, φ, z are variables in a cylindrical system of coordinates .

Quantitatively we succeed in considering only solutions (6.5) for which $X_0 \to \chi$, $X_\nu \to 0$ as $\Lambda \to 0$ (here Λ is a vector). Reckoning Λ as small, let us rewrite (6.2) in the form $X = X_0 + X_\sim$ and expand F in series with respect to the small quantities $X_\sim$; then we shall expand the result in series with respect to the harnonics X_ν $(\nu \neq 0)$, obtaining

$$F \equiv F_0(X_0) + \sum_{s=1}^{\infty} \sum_{\nu} (L_1{}^s X_{\nu_1} e^{i\nu_1\theta}) \ldots (L_s{}^s X_{\nu_s} e^{i\nu_s\theta}) = 0 \quad (6.6)$$

Here the second sum is taken for all the integers $\nu_1, \ldots, \nu_s$ not equal to zero; the matrix operators $L = L(X_0, \partial/\partial t, \partial/\partial\varphi, \partial/\partial z, \partial/\partial r, r, \lambda)$ acting on the vectors standing after them.

Now let us construct the Fourier-components of Equations (6.6), (6.2)

$$\Phi_\nu \equiv \frac{1}{2\pi}\int_0^{2\pi} F e^{-i\nu\theta}\, d\theta = 0, \qquad V_\nu \equiv \frac{1}{2\pi}\int_0^{2\pi} (UX - A)\, e^{-i\nu\theta} d\theta = 0 \quad (6.7)$$

They have the form

$$\Phi_\nu \equiv \delta_{0\nu} F_0(X_0) + (1 - \delta_{0\nu}) L_\nu X_\nu + \sum_{s=2}^{\infty}\sum_{\nu} (L_{1\nu_1}{}^s X_{\nu_1}) \ldots (L_{s\nu_s}{}^s X_{\nu_s}) = 0$$

$$V_\nu \equiv UX_\nu - A\delta_{0\nu} = 0, \qquad \delta_{0\nu} = \begin{cases} 1, & \nu = 0 \\ 0, & \nu \neq 0 \end{cases} \quad (6.8)$$

Here $L_\nu \equiv L_{1\nu}{}^1$; each of the operators L_ν is obtained from the corresponding operator L by replacing $\partial/\partial t$ by $i\omega\nu$, $\partial/\partial\varphi$ by $im\nu$ and $\partial/\partial z$ by $ik\nu$; the second sum is taken for all the numbers ν not equal to zero and satisfying the conditions $\nu_1 + \ldots + \nu_s = \nu$. In what follows Equations (6.8) will be considered for $\nu \geqslant 0$. The solution of the problem (6.8) will be sought in the form (*)

$$\omega = \omega_0 + \omega_2 q + \omega_4 q^2 + \ldots, \qquad q = QQ^*$$

$$X_\nu = Q^\nu (X_{\nu,\nu} + X_{\nu,\nu+2} q + X_{\nu,\nu+4} q^2 + \ldots), \quad X_{-\nu} = X_\nu^*, \quad \nu \geqslant 0 \quad (6.9)$$

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*) Apparently the expnsion for the frequency and the harmonics of the form (6.9) was first established in [16 and 17] for a certain actual equation containing a quadratic nonlinearity.

The choice of the expansions of ω, X_ν in the form (6.9) can be explained in the following way. The steady amplitude Q is determined to within an arbitrary phase θ_0. On the other hand, in the solution (6.5) the phase θ_0 must reduce to the form of a sum $\theta + \theta_0$, whence it follows that the harmonic X_ν is equal to the product of Q^ν with a certain function of the amplitude, not depending on the phase θ_0. The frequency ω, obviously, also cannot depend upon the arbitrary phase θ_0; this requirement is satisfied by the series in powers of q. Now we notice that if $\nu_1 + \ldots + \nu_s = \nu \geqslant 0$, then $|\nu_1| + \ldots + |\nu_s| = \nu + 2n$, where $n \geqslant 0$. Hence also from (6.9) for any term in (6.8) we obtain the estimate

$$|(L_{1\nu_1}{}^s X_{\nu_1}) \ldots (L_{s\nu_s}{}^s X_{\nu_s})| \sim |Q|^{|\nu_1| + \ldots + |\nu_s|} = |Q|^\nu q^n$$

showing that the expansion (6.9) does not contradict Equations (6.8).

Substituting (6.9) in (6.8) and collecting terms with the same powers of q, we obtain

$$\Phi_\nu \equiv Q^\nu \sum_{n=0}^{\infty} \Phi_{\nu,\, \nu+2n} q^n = 0, \qquad V_\nu \equiv Q^\nu \sum_{n=0}^{\infty} V_{\nu,\, \nu+2n} q^n = 0$$

Hence it follows that

$$\Phi_{\nu,\, \nu+2n} = 0, \qquad V_{\nu,\, \nu+2n} = 0 \qquad (\nu,\, n \geqslant 0) \tag{6.10}$$

The quantities ω_{2n}, $X_{\nu,\nu+2n}$ are determined successively from Equations (6.10). For the determination of X_{00} we have the problem

$$\Phi_{00} \equiv F_0(X_{00}) = 0, \qquad V_{00} \equiv UX_{00} - A = 0 \tag{6.11}$$

Comparison of the problems (6.3) and (6.11) shows that $X_{00} = \chi(r, \lambda)$.

The quantities X_{11} are determined from the problem

$$\Phi_{11} \equiv L_1{}^\circ X_{11} = 0, \qquad V_{11} \equiv UX_{11} = 0 \tag{6.12}$$

Here and in what follows the superscript $^\circ$ shows that the given quantity is taken when $\omega = \omega_0$, $X_0 = X_{00}$. The problem (6.12) is the problem of the theory of stability of an equilibrium state; it can have an infinite set of eigenvalues $\omega_0 = \omega_0(k, m, \lambda)$. For slight supercriticalness there is only one eigenvalue characterizing an increasing perturbation; this should be taken for ω_0 in the expansion (6.9); this eigenvalue (assumed simple) is characterized by a definite value of m (in the case [5 and 6] the value $m = 0$; in the case [7 to 10] the value $m = \pm 1$; in the case [2 and 4] the motion with $m = 0$ is fully studied [2 to 4], but there arises a steady periodic solution [4] also with $m \neq 0$). To the eigenvalue ω_0 there corresponds an eigenfunction $C_0 X_{11}$, where C_0 is an arbitrary constant, and X_{11} is a function normalized in any way. The constant C_0 remains arbitrary ; we can take $C_0 = 1$, in so far as the choice of a value $C_0 \neq 1$ is equivalent to a change of normalization of X_{11} and a related change of the amplitude Q.

In the general case the problem (6.10) for determining $X_{\nu,\, \nu+2n}$ has the form (*)

$$\Phi_{\nu,\, \nu+2n} \equiv L_\nu{}^\circ X_{\nu,\, \nu+2n} + \Psi_{\nu,\, \nu+2n} = 0, \qquad V_{\nu,\, \nu+2n} \equiv UX_{\nu,\, \nu+2n} = 0$$
$$(\nu + 2n > 1) \tag{6.13}$$

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*) The expressions $L_0(X_0)$ and $F_0(X_0)$ are connected by the relation $L_0 = \partial F_0 / \partial X_0$.

Here the functions Ψ do not depend upon $X_{\nu,\,\nu+2n}$ and contain already determined quantities.

In the Appendix it is shown that for sufficiently slight supercriticalness the homogeneous problem (6.13) with $\nu \neq 1$ does not have a solution, other than the trivial one $X_{\nu,\nu+2n} = 0$; Accordingly, the solution of the inhomogeneous problem (6.13) is [14]

$$X_{\nu,\,\nu+2n} = -\int_{r_1}^{r_2} G_\nu^\circ\,(r,\,\rho)\,\Psi_{\nu,\,\nu+2n}\,(\rho)\,d\rho \qquad (G_\nu^\circ = G\,(\nu\omega_0)) \qquad (6.14)$$

Here r_2, r_1 are boundary values of the radii, such that $r_2 \geqslant r \geqslant r_1$ (in the case of systems [5 to 10] the value of r_1 is zero); whilst the matrix operator $G\,(\omega)$ is the Green's function of the problem (6.12), in which instead of $L_1^\circ = L_1(\omega_0)$ we have the operator $L = L_1(\omega)$ (the other arguments of L_1° and L coincide).

The Green's function $G\,(\omega)$ can be represented in the form [14]

$$G = -\frac{X_{11}\,(r)\,Z^*\,(\rho)}{(i\omega - i\omega_0)\,J} + G_- \qquad \left(J = \int_{r_1}^{r_2} X_{11}{}^i Z_i{}^*\,d\rho\right) \qquad (6.15)$$

Here ω_0 is the eigenvalue of the problem (6.12) characterizing the increasing perturbation, X_{11} is the corresponding eigenfunction; $Z = \{Z_1, \ldots$ $\ldots, Z_N\}$ is the eigenfunction of the ajoint problem to (6.12), the corresponding eigenvalue being ω_0^*; the function G_- is regular when $\omega = \omega_0$ From (6.15) and (6.14) it follows that the solution of the problem (6.13) with $\nu = 1$ exists only under the condition

$$\int_{r_1}^{r_2} Z_i{}^* \Psi_{1,1+2n}{}^i\,d\rho = 0 \qquad (6.16)$$

and has the form

$$X_{1,1+2n} = -\int_{r_1}^{r_2} G_-^\circ\,(r,\,\rho)\,\Psi_{1,1+2n}\,(\rho)\,d\rho + C_n X_{11}' \qquad (6.17)$$

The condition (6.16) determines the quantity ω_{2n}. The quantities Ψ in (6.16) are given by

$$\Psi_{1,1+2n} = \omega_{2n}\,(\partial L_1 \,/\, \partial\omega)^\circ\, X_{11} + T_{1,1+2n} \qquad (6.18)$$

Here $T_{1,1+2n}$ contains quantities determined earlier. After substituting (6.18) in (6.16), we obtain

$$\omega_{2n} = -\frac{1}{J_0}\int_{r_1}^{r_2} Z_i{}^*\,T^{\,i}_{1,1+2n}\,d\rho \qquad \left(J_0 = \int_{r_1}^{r_2} Z_i{}^* \left[\frac{\partial\,(L_1)_j{}^i}{\partial\omega}\right]^\circ X_{11}{}^j d\rho\right) \qquad (6.19)$$

The quantity J_0 is different from zero; in particular, for starting equations of the form

$$\partial X \,/\, \partial t + F = 0 \qquad (6.20)$$

where F does not depend upon $\partial/\partial t$ (Equations (6.1) can usually be put into such a form by introducing supplementary unknowns), J_0 differs from $J \neq 0$ in (6.15) only by a numerical factor. The constant C_n in (6.17) remains arbitrary; we can take $C_n = 0$ in so far as the choice of a value

$C_n \neq 0$ is equivalent to a change of normalization of X_{11} (see Apendix).

It is interesting to clarify what quantities must first be calculated in order to determine $X_{\nu,\,\nu+2n}$. Let us construct Table 1 from the quantities $X_{\nu,\,\nu+2n}$ and ω_{2n}. It can be shown that in Equation (6.13) there occur all the elements of the table standing to the left of the diagonals drawn through the element $X_{\nu,\,\nu+2n+2}$. Hence it follows that in order to find ω_{2n} it is necessary first to find $X_{n+1,\,n+1}$, i.e. necessary to calculate the $(n+1)$th harmonic of the periodic motion.

TABLE 1

ν \ $\nu+2n$	0	1	2	3	4	5
0	X_{00}		X_{02}		X_{04}	
1		X_{11},ω_0		X_{13},ω_2		X_{15},ω_4
2			X_{22}		X_{24}	
3				X_{33}		X_{35}

The frequencies $\omega_{2n} = \Omega_{2n} - i\gamma_{2n}$ are complex. In so far as the frequency ω in (6.5), (6.9) must be real, then it is necessary that

$$\omega = \Omega_0 + \Omega_2 q + \Omega_4 q^2 + \ldots \equiv \Omega \tag{6.21}$$

$$\gamma \equiv \gamma_0 + \gamma_2 q + \gamma_4 q^2 + \ldots = 0 \tag{6.22}$$

Here the coefficients of the powers of q are known functions of k and λ.

7. In order to determine the wave number and amplitude of the steady periodic motion there is as yet only one Equation (6.22); the second follows from from the following hypothesis [15]: in the system a motion will become extablished of such amplitude that the maximum of the nonlinear incement γ as a function of the wave number **k** is zero (Fig.7); the value k for which the maximal increment of γ is equal to zero is also the wave number of the steady motion. This hypothesis is related to the fact that the value of q for which the maximal increment is zero is uniquely qualitatively singled out from other values of q.

According to the hypothesis thus made, the quantities k, q satisfy Equation

$$\partial\gamma/\partial k \equiv \gamma_0' + \gamma_2' q + \gamma_4' q^2 + \ldots = 0 \tag{7.1}$$

This equation shows that the hypothesis made above is equivalent to the following: in the system a motion becomes established with that wave number for which the quantity q, determined by Equation (6.22) and considered as a function of k, is maximal.

This solution of Equation (7.1) is sought in the form

$$k = k_0 + k_1 q + k_2 q^2 + \ldots \tag{7.2}$$

Substituting (7.2) in (7.1), carrying out the expansion with respect to q and equating to zero the coefficients of the powers of q, we obtain

$$\gamma_0' = 0, \qquad \gamma_0'' k_1 + \gamma_2' = 0, \ldots \tag{7.3}$$

Here the quantities γ are taken when $k = k_0$.

From (7.3) one determines one after the other the quantities k_n. The first equation determines $k_0 = k_0(\lambda)$ and shows that when $k = k_0$ the linear increment γ_0 is maximal (Fig.7). From the second equation one obtains $k_1 = -\gamma'_2/\gamma''_0$; it is immediately obvious from Fig.7 that for slight supercriticalness $\gamma''_0 \neq 0$. Similarly one finds also the other quantities $k_n = k_n(\lambda)$.

Substituting (7.2) in (6.21), (6.22) and collecting terms with the same powers of q , we obtain

$$\omega = \Omega_0 + \Omega_1 q + \ldots \qquad (\Omega_1 = \Omega_2 + \Omega_0' k_1, \ \ldots) \tag{7.4}$$

$$\gamma \equiv \gamma_0 + aq + bq^2 + \ldots = 0, \qquad (a = \gamma_2) \qquad b = \gamma_4 + \gamma_2' k_1 + {}^1/_2 \gamma_0'' k_1^2 \tag{7.5}$$

Here the quantities γ_{2n} are taken when $k = k_0$; the coefficients of powers of q in (7.4), (7,5) are known functions of the parameters λ .

The solutions of Equations (7.5) are found in Sections 1 to 5.

We notice that in systems of the second type the case of aperiodic increasing perturbations (here [2] the function $\Omega_0(k) \equiv 0$) is not in any way singled out from the point of view of applicability of the calculation; we can, however, show (see Appendix), that in this case the steady periodic solution of small amplitude does not depend on time.

8. Let us consider systems of the first type (the equilibrium state depends upon all the Cartesian coordinates, whilst perturbations and the steady motion are not periodic with respect to any of the Cartesian coordinates).

Suppose that the increasing perturbation has an oscillatory character [11]; then the calculations of Section 6 are not altered if only the quantities r, ρ are regarded as vectors (and integration with respect to ρ is carried out throughout the whole volume V , occupied by the flowing fluid), and moreover in the given case $\theta = \omega t$.

Expressions (6.5) and (6.9) show that Q occurs in the steady solution in the form of the combination $Qe^{i\omega t} = Q(t)$.

The amplitude $Q(t)$ evidently satisfies Equation

$$dQ/dt = i\omega Q \tag{8.1}$$

which retains sense even when $\gamma \neq 0$; in particular, (8.1) passes over into the equation of linear theory if we neglect in ω all powers of q . If in (8.1) we set $Q = |Q|e^{i\theta}$ and separate the real and imaginary parts, then we obtain (1.1) and Equation $d\theta/dt = \Omega$.

In the case of systems of the first type the observed steady periodic motion always correspond to the stable solutions of Equation (1.1) (in which γ is taken in the form (6.22)).

In the case of systems of the second type steady motion becomes established as a result of the interaction of a continuous spectrum of increasing waves. In the study of stability of steady solutions of Equation (1.1) for the divergence δq from the steady value of q we obtain

$$d\delta q/dt = \delta q\,(\gamma + q\partial\gamma/\partial q + q\,(\partial\gamma/\partial k)dk/dq) \tag{8.2}$$

Here $\gamma = \gamma(k, q)$ is defined in (6.22), and $k = k(q)$ in (7.1), (7.2).

By virtue of (6.22) and (7.1), there remains in (8.2) only the second term, in which k is equal to the wave number of the steady solution (7.2); hence it follows that the stability of the steady solution with the wave number (7.2) is studied only with respect to the perturbation δq with the same wave number. Accordingly, study of the stability on the basis of Equation (1.1) is not in the given case complete (in contrast to the case of systems of the first type), and the observed steady motions correspond to the stable steady solutions of (1.1) only when the former really are periodic

Now let us consider systems of the first type which for slight supercriticalness are unstable with respect to aperiodic perturbation. It is to be expected that the steady solution in this case does not depend on time and is determined completely (it does not contain an arbitrary phase). It can be assumed that $\chi = A = 0$ for problems (6.1), (6.2) (this can always be achieved by the introduction of a new unknown $X_* = X - \chi$) . We seek the solution in the form

$$X = QX_1 + Q^2X_2 + \ldots, \partial / \partial t = \gamma \equiv \gamma_0 + Q\gamma_1 + Q^2\gamma_2 + \ldots \qquad (8.3)$$

Here Q is the real amplitude; it is convenient to take $Q > 0$.

Substituting (8.3) in (6.1), (6.2) and equating to zero the coefficients of powers of Q , we obtain the problems for the determination of γ_{n-1} and X_n .

When $n = 1$ we obtain the linear problem of the theory of stability

$$L^\circ X_1 = 0, \qquad UX_1 = 0 \qquad (L = L(\gamma, \nabla, r, \lambda)) \qquad (8.4)$$

Here and in what follows the superscript $^\circ$ indicates that the corresponding quantity is taken when $\gamma = \gamma_0$. For γ_0 and X_1 in (8.3) one should take the eigenvalue (assumed simple) and the eigenfunction of the problem (8.4) which characterize the increasing perturbation; according to the condition $\gamma_0 > 0$ and therefore X_1 can be assumed real. When $n > 1$ we have the problem

$$L^\circ X_n + \gamma_{n-1} (\partial L / \partial \gamma)^\circ X_1 + T_n = 0, \qquad UX_n = 0 \qquad (8.5)$$

Here T_{11} depends on quantities determined earlier. Let $G(\gamma)$ be the Green's function for the problem (8.4); it is obtained from (6.15) by replacing X_{11} by X_1 and $i\omega$ by γ . From (8.4) and (6.15) it follows that the solution of the problem (8.5) exists under the condition

$$\gamma_{n-1} = -\frac{1}{J_0} \int_V ZT_n d\rho \qquad \left(J_0 = \int_V Z \frac{\partial L}{\partial \gamma} X_1 d\rho\right) \qquad (8.6)$$

and has the form (8.7)

$$X_n = -\int_V G_-^\circ (r, \rho)\, \Psi_n (\rho)\, d\rho + C_{n-1}X_1, \qquad \Psi_n = T_n + \gamma_{n-1} \left(\frac{\partial L}{\partial \gamma}\right)^\circ X_1$$

Here Z is the real eigenfunction of the adjoint problem (8.8), corresponding to the value $\gamma_0 > 0$. The constants C_n are arbitrary; we can set

$C_n = 0$ (and the normalization of the function X_1 is unchanged).

To the observed motions correspond the stable steady solutions $Q > 0$ of Equation $dQ / dt = \gamma Q$, where γ is determined from (8.4). It can be shown (see Appendix) that γ is real for sufficiently small values of Q.

It is to be noted that the nonlinear increment γ can be calculated by a method differing from those described in Sections 6 and 8 (see Appendix)

All that was said above concerning abrupt transition is applicable to systems with a finite number of degrees of freedom (described by the ordinary differential equations (6.1)). In this case Equations (6.10) are algebraic, and the problem of finding the quantities ω_{2n}, $X_{\nu,\nu+2n}$ simplifies so much as to make possible the consideration of actual examples [18] (without the application of computers).

Appendix. We shall show that the homogeneous problem (6.13) with $\nu \neq 1$ does not have nontrivial solutions. It is sufficient to prove the assertion for $\Lambda = 0$; then it remains true also for sufficiently small Λ, in so far as $q \to 0$ when $\Lambda \to 0$ (here Λ is a vector). According to the definition of the critical parameters λ_*, the linear increment $\gamma_0 = \gamma_0(k, \lambda_*)$ vanishes when $k = k_0(\lambda_*) = k_*$ and is negative when $k \neq k_*$ (Fig.7). Moreover, the frequency $\omega_0 = \omega_0(k, \lambda_*)$ is real when $k = k_*$ (and equal to ω_*) and complex when $k \neq k_*$. Accordingly, problem (6.13) with the operator $L_* = L_1^\circ(\lambda_*)$ has a real eigenvalue $\omega_0 = \omega_*$ only when $k = k_*$; when $k = \nu k_*$ ($\nu \neq 1$) the eigenvalues ω_0 are complex, and consequently the real value $\omega_0 = \nu\omega_*$ is not an eigenvalue.

We shall show that the choice of the constants $C_n \neq 0$ in (6.17), (8.9) is equivalent to a change of normalization of X_{11} and X_1. Let q be the amplitude of the steady solution corresponding to the choice $C_0 = 1$, $C_1 = C_2 = \ldots = 0$. Let us introduce the "new" amplitude $Q^\wedge$ by Equation

$$Q = C\,(Q / C) \doteq C\,(Q^\wedge) \tag{A.1}$$

If we take C in the form

$$C = C_0{}^\wedge + C_1{}^\wedge q^\wedge + C_2{}^\wedge (q^\wedge)^2 + \ldots, \quad q^\wedge = Q^\wedge Q^{\wedge *}$$

substitute (A.1) in the solution of (6.5) and (6.9) and collect terms with identical powers of $Q^\wedge$, then we obtain expressions depending on the constants $C_0{}^\wedge$, $C_1{}^\wedge$, ... ; they can be selected so as to obtain the solution of (6.9) with arbitrary values of the constants C_0, C_1, ... A similar transformation of the solution of (6.9) is obtained if instead of (A.1) we take $X_{11} = C\,(X_{11} / C) = C X_{11}{}^\wedge$ in Expressions of $X_{\nu,\,\nu+2n}$ in terms of X_{11}.

We shall show that if in (6.9) the quantity $\Omega_0 \equiv 0$, then also $\Omega_{2n} = 0$, i.e. the steady periodic solution does not depend on time. Let us seek the solution X in the form of a real Fourier series with respect to the spatial coordinates, in which the nth harmonic is proportional to $\exp n\gamma t$. The coefficients of the series and the increment γ will be sought in the form of an expansion of type (6.9) with respect to the real amplitude Q. Moreover, to determine γ_{2n} and the quantities of type $X_{\nu,\,\nu+2n}$ we obtain real equations (in so far as the starting problem (6.1), (6.2) is real). According to the condition, γ_0 is real, and therefore the functions of type X_{11} can be taken real; $G(\gamma)$ is also real for real γ, and therefore γ_{2n} and the quantities of type $X_{\nu,\,\nu+2n}$ are obtained real. In the case of the problem (8.3) to (8.7) the quantities γ_n are also real.

The steady solutions can be found by the method of [17], in which the dependence on time is completely included in the amplitude Q; in the case of periodic solutions for Q we postulate Equations (8.1), (6.9), and seek a solution for X in the form

$$X = \sum_{n=0}^{\infty} X_n \qquad (|X_n| \sim |Q|^n) \tag{A.2}$$

It is appropriate to take $X_0 = \chi$ and

$$X_1 = QX_{11} + (QX_{11})^* \qquad (A.3)$$

Then for the real quantities X_n we obtain

$$X_n = \sum_{\nu=0}^{n} Q^{n-\nu} (Q^*)^\nu X_{n-2\nu, n}, \qquad X_{-\nu, n} = X_{\nu, n}^* \qquad (A.4)$$

For simplicity we shall assume that the starting equations have the form (6.20); then on substituting (A.4), (A.2) in (6.20) we obtain with the help of (8.1)

$$d(Q^\nu q^n)/dt = Q^\nu q^n (i\nu\omega + 2n\gamma) \qquad (\omega = \Omega - i\gamma) \qquad (A.5)$$

Here γ, ω are, respectively, the nonlinear increment and frequency. If in the expansions (6.20), (6.2) with respect to powers of Q we equate to zero the coefficients of $Q^\nu q^n$, then we obtain the problem for the determination of $X_{\nu, \nu+2n}$. When $\nu = 1$, $n = 0$ we have the problem of the theory of stability (6.12). When $\nu + 2n > 1$ we obtain the problem (6.13), in which $L_\nu^\circ = L_1 (i\nu\omega_0 + 2n\gamma_0)$; if $\gamma_0 \neq 0$, then for any ν the solution has the form (6.14). It is not difficult to see that $X_{1, 1+2n} \to \infty$ as $\Lambda \to 0$, since the denominator of the first term in Expression (6.15) for C is proportional to $2n\gamma_0$. For boundedness [17] of the quantities $X_{1, 1+2n}$ when $\Lambda = 0$ it is necessary that the condition (6.16) be fulfilled, from which we find ω_{2n} from (6.19). Moreover for $X_{1, 1+2n}$ we obtain Expression (6.17), in which $C_n = 0$.

In the case of aperiodic steady motion it is appropriate to postulate for Q that $dQ/dt = \gamma Q$; where γ, X are chosen in accordance with (8.3) and (8.4). In this case

$$dQ^n/dt = Q^n (\gamma + (n-1)\gamma) \qquad (A.6)$$

Then problem (6.20), (6.2) is solved just as in the case of periodic steady motion.

We note that for solution of the problem (6.20). (6.2) by the method described in Sections 6 and 8, we take into account only the first terms in the left-hand sides of (A.5) and (A.6). In both methods, however, the expressions for the derivatives (A.5), (A.6) have one and the same physical meaning when $\gamma = 0$; hence it follows that the solutions obtained by the two methods are physically identical and differ only in the normalization of the functions X_{11} and X_1.

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